As a freshman at the university I was troubled to learn that scientists had no idea at all about why the forces of nature act as they do; they could only describe nature's forces by experimental results and tabulate the results in mathematical equations. The often quoted, "a wise man knows he knows nothing at all" came to mind.

Later in my studies, I became amazed at the elegant simplicity of the mathematical equations that could not only accurately describe the results of many experiments, but could also accurately predict new results unknown when the equations were first derived. The most amazing is the Schrödinger equation, the foundation of quantum mechanics; it correctly predicts the results of experiments involving photons, electrons, protons, nuclei, atoms, and molecules. But even more exciting, quantum mechanics has unveiled the non-locality, entanglement and chance inherent in our universe; physics is once again united with philosophy.

### The Force of Gravity

The first of Mother Nature’s veils was removed in the 1600s; why do the planets move the way they do? Tradition has it that Sir Isaac Newton was walking in his garden when an apple fell; he suddenly understood that the apple falls down rather than up or sideways because there is a force between the apple and the earth directed toward the center of the earth. He recognized that there is a natural force attracting any two objects to each other.

By studying Kepler's mathematical descriptions of planetary motion and results of dropping metal balls from various heights, he concluded that there exists an attractive force between any two objects that is equal to the product of their masses times a universal constant (the gravitational constant G) divided by the square of distance between them

\[ F_g = \frac{m_1 m_e G}{d^2} \]

**Newton’s Law of Gravitation**

\[ a_g = \frac{m_e G}{r^2} \]

*Fₕ* is the force on an object with mass *m₁* due to the earth's gravity, *d* is the distance between the object and the center of the earth, and *mₑ* is the mass of the earth. When *r* equals the radius of the earth, *aₕ* equals the acceleration (of any mass) due to gravity at the earth’s surface; *aₕ* = 9.8 meters per second per second. He believed that the force of gravity exists between any two particles in the universe, no matter how far apart.
Newton was the first to formulate laws of mechanics. His three famous laws are: (1) An object at rest stays at rest, and an object in motion moves in a straight line with constant speed unless acted on by a force (a reference frame wherein the first law is true is often called an inertial reference frame). (2) In any reference frame where the first law is true, the rate of change of momentum is proportional to the impressed force, and is in the direction in which the force acts; \( F = \frac{dp}{dt} \) (\( p = mv \) where \( v \) is the velocity of an object of mass \( m \), \( p \) is its momentum, and \( d/dt \) is the math symbol for rate of change with respect to time). (3) To every action there is an equal and opposite reaction.

Newton discovered that mechanical problems are most easily expressed in terms of the concept of force as he defined above. He believed that if the laws hold in a particular inertial reference frame, then they will also hold in any other inertial reference frame -- the principle of Newtonian Relativity.

**Special Relativity**

In 1887 Michelson and Morley conducted a series of experiments that suggested the speed of light in free space \( c \) is the same everywhere, regardless of any motion of the light source or the observer (\( c = 300 \text{ million meters per second} \)). For example, if a man standing at train station fires a flare at the exact instant that another man in a train going 150 km/hour is passing, both men will be at the center of the expanding sphere of light! This implies that the reference frame of each observer, from the point of view of the other, is being affected by their relative motion. The effects defy common sense.

In 1892, Hendrik Lorentz showed how the coordinate frames of two observers would need to be related mathematically when the speed of light has the same value in both frames:

\[
\begin{align*}
x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\
y' &= y \\
z' &= z \\
t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}
\end{align*}
\]

**Lorentz Transformation**

Superscript ‘ denotes the frame of an observer moving in the x direction with constant speed \( v \) relative to an observer at rest. To the moving observer an event occurs at location \( x', y', z' \) at time \( t' \). To the stationary observer the same event occurs at location \( x, y, z \), and at time \( t \). The speed of light has the same value \( c \) in both frames.

The Lorentz transformation has some unusual implications: If we measure the length of a car to be \( L_0 \) when it is parked, we will measure its length to be \( L_0\sqrt{1-v^2/c^2} \) when it is moving past us with the velocity \( v \) (Lorentz-FitzGerald Contraction). The length of an object
is a maximum when measured in a reference frame in which it is stationary. A stationary clock measures a longer time interval between events occurring in a moving frame of reference than does a clock in the moving frame (Time Dilation). Two events that occur simultaneously at two different places (i.e., two different values of x) to a stationary observer may not occur simultaneously to another observer moving relative to the first.

In 1905 Albert Einstein presented his theory of special relativity wherein: the speed of light c has the same value for all observers, regardless of their motion; and “the laws of physics [e.g., conservation of momentum, F=dp/dt] may be expressed in equations having the same form in all reference frames moving at constant velocity with respect to one another”.

Special Relativity expands the Lorentz transformations:

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

$$p = \frac{m_0v}{\sqrt{1-v^2/c^2}}$$

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = \frac{d\left[m_0v/\sqrt{1-v^2/c^2}\right]}{dt}$$

$$T = \int F ds = mc^2 - m_0c^2$$

$$E = mc^2$$

$$E = T + m_0c^2$$

$m_0$ is the mass of a body at rest relative to an observer, $m$ is the mass of the same body moving at speed $v$ relative to the observer, $p$ is the momentum of the body moving at speed $v$ relative to the observer (i.e., $v$ is the speed between the observer and whatever he is observing), $T$ is the kinetic energy of the body (i.e., the work done in bringing it from rest to its velocity $v$; the integral of force times distance), $E$ is defined as the total energy of a body of rest mass $m_0$ moving at speed $v$ relative to the observer, $m_0c^2$ is defined as the "rest energy" of a body at rest relative to the observer, and $c$ is the speed of light.

The equation for total energy $E$ implies that even when a body is at rest it never-the-less posses the energy $m_0c^2$ (e.g., $m_0$ of 1 kg has a rest energy content of $9 \times 10^{16}$ joules - billions of kWh)! "Even a minute bit of matter represents a vast amount of energy, and, in fact, the conversion of matter into energy is the source of the power liberated in all exothermic reactions of physics and chemistry . . . mass can be created or destroyed, but only if an equivalent amount of energy simultaneously vanishes or comes into being, and vice versa."

For example, the mass (energy/c^2) of the chemical bonds in carbon dioxide is one ten-billionth of the total mass of the molecule, so it is not surprising that a measurement of the sum of the masses of the atoms may appear to equal the mass of the molecule. On the other hand, in an atomic nucleus, the binding energy contributes anywhere from 0.1% up to about 1% of the total mass of the nucleus, so in nuclear reactions we can more easily observe the conversion of mass to energy.
When the velocity, \( v \), is much less than the speed of light (i.e., \( v^2/c^2 << 1 \)), the Relativistic equations reduce to the equations of Newtonian mechanics: \( m=mv_0 \), \( p=m_0v \), and \( T=m_0v^2/2 \). Newton's original equations are used by engineers today to put satellites into orbit, send rockets to the moon, and to solve everyday mechanical problems here on earth.

**Electromagnetic Force**

The second veil was removed in the mid 1800s. Why does a glass rod rubbed with silk move toward a hard rubber rod rubbed with cat's fur? Why does an electric current in a coil of wire cause a nearby magnetized needle to be deflected? Why does simply moving a magnet closer or farther away from a wire loop induce an electrical current in the loop? Why does an alternating current in a wire induce an alternating current in a distant wire?

In 1785 Charles-Augustin de Coulomb discovered that there exists a force, \( F_c \), between two electrostatic charges, \( Q_1 \) and \( Q_2 \), equal to the product of the charges times a constant \( K \) divided by the square of the distance between them:

\[
F_c = \frac{KQ_1 Q_2}{d^2}
\]

**Coulomb's Law**

In General, when \( Q_2 \) is replaced by many different charges at many different locations, we write \( F_c = Q_1 \mathbf{E} \) where \( \mathbf{E} \) is called the Electrostatic Field.

[We now know that the sources of electrical charges are the protons (positive charges) and the electrons (negative charges) found in atoms. Normally bodies contain protons and electrons in equal numbers. Processes, such as rubbing a glass rod with silk, transfer electrons from one body to another, leaving one body positively charged (electron deficiency) and the other negatively charged (electron excess)].

In addition to the gravitational force due to their masses, any two charged particles in the universe will also be attracted or repelled due to the Coulomb force, no matter how far apart. The Coulomb force between an electron and a proton is \( 10^{36} \) times greater than the gravitational force between them.

In 1825 Andre-Marie Ampère demonstrated that there is a force, \( F_m \), between two electrical circuits, \( C_a \) and \( C_b \), carrying electric currents, \( I_a \) and \( I_b \). The force is equal to the product of the currents times a function that depends only on the geometries of the circuits. In the special case of long parallel wires, the force per meter \( F_m/m \) between the wires equals the constant \( \mu_0/2\pi \) multiplied by the product of the two currents divided by the distance \( d \) between the wires.
wires. The force is attractive if the currents flow in the same direction and repulsive if they flow in opposite directions. For long parallel wires:

\[ F_{\text{m}}/m = (\mu_0/2\pi) \frac{l_1 l_2}{d} = I_a B_b \quad \text{where} \quad B_b = \frac{\mu_0 I_b}{2\pi d} \]
\[ F_{\text{m}}/m = (\mu_0/2\pi) \frac{l_1 l_2}{d} = -I_b B_a \quad \text{where} \quad B_a = \frac{\mu_0 I_a}{2\pi d} \]

\( B_b \) is called the Magnetic Field produced by the current \( I_b \), and \( B_a \) is called the Magnetic Field produced by the current \( I_a \). (Note: if the direction of wire b from wire a is positive, then the direction of wire a from wire b is negative; the force on wire a due to the magnetic field produced by wire b is equal and opposite to the force on wire b due to the magnetic field produced by wire a; Newton's third law is satisfied). Ampère speculated that the force between two electrical currents was the same as the force between two magnets; "the magnetic force of magnets is due to perpetually flowing loops of current within the magnets" (verified 100 years later with quantum mechanics).

Combing the Coulomb force on a charge \( Q \) in the presence of an electric field \( E \) with the magnetic force on \( Q \) when \( Q \) is moving with velocity \( u \) in the presence of a magnetic field \( B \) gives the total force on \( Q \):

\[ f = Q \left[ E + (u \times B) \right] \quad \text{Lorentz force} \]

The cross product \( u \times B \) is a vector perpendicular to both \( u \) and \( B \) (right hand rule). Its magnitude equals the magnitude of \( u \) times the magnitude of \( B \) times the sine of the angle between them.

In 1831, Michael Faraday demonstrated that a changing magnetic field induces an electromotance (e.g., a voltage) in a circuit, and that moving a circuit in a constant magnetic field also generates an electromotance in the circuit.

Between 1861 and 1865, James Clerk Maxwell developed four simple equations which described all of the experimental results of electromagnetism. Maxwell's equations in free space:

\[ \nabla \cdot E = \rho/\varepsilon_0 \quad \nabla \cdot B = 0 \quad \nabla \times E = -\partial B/\partial t \quad \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \partial E/\partial t \quad \text{Maxwell's Equations} \]

\( E \) is the electrostatic field, \( B \) is the magnetic field, \( \rho \) is the electric charge per unit volume (which may depend on time and position), \( \varepsilon_0 \) is the electric constant, \( \mu_0 \) is the magnetic constant, \( J \) is the electric current per unit area (which may depend on time and position), and \( c^2 = 1/\mu_0 \varepsilon_0 \) where \( c \) equals the speed of light in free space. The equations are expressed using standard mathematical operators: \( \nabla \cdot \) (divergence; sink or source), \( \nabla \times \) (curl; circulation), \( \partial \) (partial derivative).
Maxwell's Equations show that an electrical charge and/or a magnetic field that is changing in time will induce an electric field, and an electrical current and/or an electric field that is changing in time will induce a magnetic field. The equations are already compatible with Einstein's Theory of Special Relativity; the speed of light $c$ and the equations themselves are the same in all inertial reference frames. In fact, Maxwell equations were the inspiration for Einstein's Theory.

Solutions to these equations include self-sustaining electromagnetic "waves" traveling at the speed of light. Maxwell showed that light itself is just such an electromagnetic wave. In regions with no charges ($\rho = 0$) and no currents ($\vec{J} = 0$), such as in a vacuum, Maxwell's equations reduce to standard wave equations:

$$\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} - \nabla^2 E = 0$$
$$\frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} - \nabla^2 B = 0$$

$\nabla^2$ is the Laplace operator which is shorthand for $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. There are an infinite number of solutions to a wave equation but only certain solutions will satisfy the initial/boundary conditions, known values of $\vec{B}$ and $\vec{E}$ at certain times and/or certain locations.

The term "wave" may be misleading. Unlike a water wave where the water moves up and down as the wave passes and an air wave where the air compresses and relaxes as the wave passes, electromagnetic waves (i.e., light waves) can travel in a vacuum where there is no medium. The term "wave" was popularized in 1801 when Thomas Young observed that a light beam going through a double slit creates an interference pattern exactly like those of air waves and water waves. For the next 100 years physicists searched for light's medium, the "luminiferous ether," without success.

Since the speed of light has the same value in all inertial frames, the Lorentz transformation must be used to compare the electric and magnetic fields observed in one frame with those observed in a second frame moving with velocity $\vec{v}$ relative to the first. For example, the primed frame is moving relative to the unprimed frame at velocity $\vec{v}$ in the $x$ direction, i.e., velocity vector $\vec{v} = (v, 0, 0)$. The electric and magnetic fields are vector fields, $\vec{E} = (E_x, E_y, E_z)$ and $\vec{B} = (B_x, B_y, B_z)$:

$$y = 1/ [1 - v^2/c^2]^{1/2}$$
$$\vec{E}' = y (\vec{E} + \vec{v} \times \vec{B}) - (y - 1) E_x \quad \text{Lorenz Transformation}$$
$$\vec{B}' = y (\vec{B} - \vec{v} \times \vec{E}/c^2) - (y - 1) B_x$$

Maxwell's equations and the Lorentz Force have the same form in both the primed and unprimed frames, but the Lorentz transformation shows that the values of the Magnetic
and Electric fields may be different to observers in different frames. When the relative velocity of the reference frames is small relative to the speed of light $\gamma \approx 1$ (i.e., $v^2/c^2 << 1$)

$$E' = E + v \times B$$
$$B' \approx B - \frac{v \times E}{c^2}$$ Non-Relativistic Transformation
$$J' = J - \rho \frac{v}{c^2}$$
$$\rho' = \rho - \frac{J \cdot v}{c^2}$$

The electric field and magnetic fields are really two interrelated aspects of a single electromagnetic field:

"If a conductor moves with a constant velocity through the field of a stationary magnet, eddy currents will be produced due to a magnetic force on the electrons in the conductor. In the rest frame of the conductor, on the other hand, the magnet will be moving and the conductor stationary. Classical electromagnetic theory predicts that precisely the same microscopic eddy currents will be produced, but they will be due to an electric force." (1905 Einstein)

Maxwell's equations not only correctly predicted the results of all the experiments at the time but also the results of future electromagnetic experiments until today. They are still used to solve problems involving nature's electromagnetic force. Telephones, generators, transformers, electric motors, telegraph, radio, television, telecommunications, antenna, electrical power grids, electro-optical systems etc. are designed with Maxwell's equations.

**Quantum Mechanics**

The third veil was removed in the early 1900s. Why does a metal rod change color from dull red to bright orange-red to white hot as it is heated to progressively higher temperatures? Why are electrons emitted from a metal surface only when certain high frequency light falls on it (the photoelectric effect)? Why does light sometimes behave as a stream of particles? Why do rarefied gases excited by an electric current only radiate light of certain frequencies (discrete spectral lines) that are characteristics of the gas used? And why do all the spectral lines split in the presence of a magnetic field? Why do atoms exhibit the chemical properties summarized by Mendeleev's Periodic Table? Why do electrons exhibit a wave-like diffraction pattern when they are scattered from certain crystals? Why do measurements made during an experiment always affect the experiment's results? Why do particles sent through double slits form the classic wave interference pattern? How can the force of gravity adjust instantaneously when two masses change their relative position?
In 1900 Max Planck explained the changing colors of the radiation emitted from objects heated to higher and higher temperatures. He derived a formula that accurately predicted the intensity of the observed colors by assuming that light was emitted in small, separate, bursts of energy. The small bursts are called "quanta." He formulated that the quanta associated with a particular frequency of light, \( \nu \), all have the same energy

\[
E = h\nu
\]

He explained that the energy of the quanta, \( \nu \) is the frequency of the light, and \( h = \text{Planck's constant} \) \( (6.625 \times 10^{-34} \text{ joule-sec}) \).

In 1905 Albert Einstein explained the photoelectric effect based on Planck's quanta. He showed that light interacts with matter as individual quanta (called photons). His Photoelectric effect formula is

\[
h\nu = T_{\text{max}} + h\nu_0
\]

\( h\nu = \text{the energy of each quantum of the incident light} \), \( T_{\text{max}} = \text{the maximum photoelectron energy} \), and \( h\nu_0 = \text{the minimum energy needed to dislodge an electron from the metal surface being illuminated} \). The quantum theory of light predicts correctly that the maximum electron energy depends on the frequency of the incident light and not upon its intensity. This explains why even the feeblest light of the right frequency can lead to the immediate emission of photoelectrons. Einstein's "photon of light" has energy \( h\nu \), zero rest mass, travels at the speed of light \( c \) and has momentum \( p = h\nu/c \). He proved that light has properties of particles!

In 1924 Louis de Broglie proposed that not only do light waves have properties of particles but also that particles have properties of waves. He calculated the wavelength of a material particle using analogy with photon wavelength. For a photon with momentum \( p \), \( p = h\nu/c = h/\lambda \) where \( \lambda \) is the wavelength of the light. For a material particle of mass \( m \) and velocity \( v \), de Broglie wavelength \( \lambda \) is given by

\[
\lambda = h/p = h/mv
\]

de Broglie wavelength

The relativistic mass \( m = m_0/[1-v^2/c^2]^{1/2} \).

De Broglie associates a moving particle with a wave group. Mathematically, a wave group is a series of individual waves with slightly different wavelengths; their interference with each other results in the variation of amplitudes exhibited by the de Broglie wave group Fig 1. Think of tuning a guitar; when two strings have nearly the same frequency you will hear beats. If the speeds of the individual waves producing the beats depend on their
wavelengths, then the individual waves do not travel together; the speed of the beats is
different from that of the individual waves that compose it. The de Broglie wave group
c Travels with a group velocity $v$, the same velocity as the particle.

![Figure 1  de Broglie Wave Group](image)

The wave property of material particles was demonstrated in 1927. The Davisson-Germer
experiment used a single large crystal to show that a stream of electrons produced a wave-
like diffraction pattern consistent with their de Broglie wavelength. The wavelength of a 50
eV (electron-volt) electron is on the same order of magnitude as the lattice spacing of a
 crystal, but the wave length of a car travelling at 100 km/hour is on the order of $10^{-38}$
meters, far too small to be detected.

But what is it that waves? What is it that varies in space and time for a de Broglie wave?
The "it" is an abstract mathematical function $\Psi(x, y, z, t)$ called the wave function; $\Psi^2$
equals the probability of experimentally finding the particle at the point $x, y, z$ at the time $t$. If an
experiment involves a large number of identical particles, then the actual density of
particles at point $x, y, z$ at time $t$ is given by $\Psi^2$ where $\Psi$ is the wave function that describes
all the particles. As we shall see, $\Psi$ is a fundamental mystery of quantum mechanics.

Also in 1924, Werner Heisenberg formulated his famous Uncertainty Principle, "at any
instant in time, the product of the uncertainty in the location of a particle with the
uncertainty in the particle's momentum is always greater than or equal to Planck's
constant;"

$$\Delta x \Delta p \geq \hbar$$  Heisenberg's Uncertainty Principle

In other words, it is impossible to measure simultaneously both position and momentum of
a particle with perfect accuracy. If the de Broglie wave group is very narrow then the
particle's position (i.e., the center of the wave group) at any instant can be determined but
not its wavelength $\lambda$ or momentum $p$ ($p=\hbar/\lambda$). If the wave group is very broad then its
wavelength and momentum can be determined but not its position. From a measurement
perspective, the more accurately we wish to measure the position of a particle the shorter
the wave length of the photon we must use to measure it, but the shorter the photon's
wave length, the greater the momentum the photon will transfer to the particle in
measuring its position.
**Schrödinger Wave Function**

In 1926 Erwin Schrödinger was aware of Planck's and Einstein's work on the particle property of waves, and Heisenberg's work on the uncertainty principle. He had worked together with de Broglie on formulating the wave properties of particles, in particular the wave function $\Psi$. He also knew that a wave equation could, depending on the boundary conditions, have solutions with discrete wave frequencies like Planck's quanta.

Schrödinger's problem was to integrate all these new ideas and facts to determine the possible states of a particle, such as its energy and angular momentum, in the presence of a force such as the coulomb or the gravitational force. In particular, he wanted to determine the possible quantum states of the electron in a hydrogen atom bound by the attractive coulomb force of the nucleus and thereby explain the observed gas spectral lines of hydrogen and radiations in general.

He realized that no solution could simultaneously give the exact position and momentum of the electron because of Heisenberg's uncertainty principle. Instead he focussed on finding the electron's wave function $\Psi$. The probability that a particle is located between $x_1$ and $x_2$ is given by the integral

$$ \int_{x_1}^{x_2} \psi^2 dx $$

(If $x_1=-\infty$ and $x_2=+\infty$ the integral is equal to 1)

In order to obtain a wave equivalent of a bound electron of total energy $E$ and momentum $p$, Schrödinger used the de Broglie equation for wave length of a particle, $\lambda = h/p$, and Planck's equation for frequency of a photon, $\nu = E/h$ ($\nu\lambda=c$). After several years, the results of his thinking produced the Schrödinger equations for $\Psi$ and $\psi$:

$$ \left(\frac{\hbar}{2\pi i}\right)\frac{\partial \Psi}{\partial t} = \left(\frac{\hbar^2}{8\pi^2 m}\right) \nabla^2 \Psi - V\Psi $$  \hspace{1cm} **Schrödinger Equation (Time-dependant)**

$$ \nabla^2 \psi + \left(\frac{8\pi^2 m}{\hbar^2}\right)(E - V)\psi = 0 $$  \hspace{1cm} **Schrödinger Equation (Time- independent)**

$E$ is the total energy of the particle; $E = T + V$ where $T$ is the kinetic energy of the particle, $T = p^2/2m$. $V$ is the particle's potential energy due to the Coulomb force, $m$ is the mass of the particle, $\hbar$ equals Planck's constant and $i = [-1]^{1/2}$. If the potential energy $V$ of the particle does not depend on time $t$ explicitly, the force that acts on the particle varies only with the particle's position:

$$ \Psi = \psi e^{-\frac{2\pi i E}{\hbar} t} $$  \hspace{1cm} **Time-independent Forces**

For the case of an electron with charge $-e$ bound to a hydrogen atom with nucleus of one proton with charge $+e$, the potential energy $V$ is just the electrostatic, Coulomb force potential energy of the electron where $r$ is the distance between electron and proton:
The potential $V$ does not depend on time $t$ so we need only solve the time-independent Schrödinger equation for $\psi$. Note, $\psi$ is a function of the coordinates $x$, $y$, $z$ with origin $0$, $0$, $0$ at the proton. The distance from the origin is given by $r = \left( x^2 + y^2 + z^2 \right)^{1/2}$.

Because of the symmetry of the problem, it is easier to solve the time independent Schrödinger equation when it is expressed in spherical coordinates $(r, \theta, \phi)$ rather than in Cartesian coordinates $(x, y, z)$; $\psi = \psi(r, \theta, \phi)$, $r$ is the radial distance to a point, $\theta$ is its latitude, and $\phi$ is its longitude. The Schrödinger equation in spherical coordinates with the electrostatic potential $V$ defined above is

$$\frac{\sin^2 \theta}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{\partial^2 \psi}{\partial \phi^2} + \left( \frac{8\pi^2 m^2 \sin^2 \theta}{h^2} \right) \left( \frac{e^2}{4\pi \varepsilon_0 r} + E \right) \psi = 0$$

By expressing the Schrödinger equation in spherical coordinates, the solution can be found using the method of separation of variables wherein $\psi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi)$. Substituting $R\Theta\Phi$ into the above equation yields three decoupled ordinary differential equations:

$$\frac{d^2 \Phi}{d\phi^2} + m_l^2 \Phi = 0$$

$$\left( \frac{1}{\sin \theta} \right) \frac{d}{d\theta} \left( \sin \theta \frac{d \Theta(\theta)}{d\theta} \right) + \left[ l(l+1) - m_l^2 / \sin^2 \theta \right] \Theta = 0$$

$$\left( \frac{1}{r^2} \right) \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \left[ \left( \frac{8\pi^2 m}{h^2} \right) \left( \frac{e^2}{4\pi \varepsilon_0 r} + E \right) - l(l+1) / r^2 \right] R = 0$$

We find that solutions for $\psi$ exist if and only if:

1) The total energy $E$ has one of the values $E_n = - \left( \frac{me^4}{8\varepsilon_0^2 h^2} \right) \left( \frac{1}{n^2} \right)$ where $n$, called the "Total quantum number," has one of the values $n = 1, 2, 3, \ldots$

2) The "Orbital quantum number," $l$, has one of the values $l = 0, 1, 2, \ldots, (n - 1)$

3) The "Magnetic quantum number," $m_l$, has one of the values $m_l = 0, \pm 1, \pm 2, \ldots, \pm l$.

4) The "Electron Spin quantum number", $m_s$, has one of two values $m_s = +1/2$ or $-1/2$.

The total quantum number $n$ describes the quantization of the electron's total energy. The fact that each of its total energy states is negative merely means that electron is bound to the atom by the Coulomb force; unbounded electrons are not quantized. The quantum number $l$ describes the quantization of the magnitude of the electron's angular momentum vector, $L = \hbar \left( \frac{l(l+1)}{2\pi} \right)^{1/2}$. The direction of $L$ is perpendicular to the plane of motion of the electron. Like the total energy $E$, angular momentum is both conserved and quantized.

If the atom is placed in an external magnetic field, $B$, there will be a torque on the electron's angular momentum vector that can change its direction. The Magnetic quantum numbers, $m_l$, describe the permissible angles $\delta$ between $B$ and $L$, cosine $\delta = m_l / \left| l(l+1) \right|^{1/2}$;
"We must regard an atom characterized by a certain value of \( m_l \) as standing ready to assume a certain orientation of its angular momentum, \( L \), relative to an external magnetic field in the event it finds itself in such a field."

"The total energy state of an electron with total quantum number \( n \) breaks up into several substates when the atom is in a magnetic field, and their energies are slightly more or slightly less than the energy of the state in the absence of the field." (1963 Arthur Beiser)

Using special relativity, Paul Dirac showed, in 1927, that there is always an interaction between an electron's "intrinsic spin" and its orbital angular momentum. The total energy state of the electron with total quantum number \( n \) will be slightly higher (\( m_s = \frac{1}{2} \)) or slightly lower (\( m_s = -\frac{1}{2} \)) due to spin-orbit coupling than the energy value shown in 1) above.

The results of the Schrödinger equations have an analogy in planetary motion; the inverse square law of the gravitational force has the same form as the inverse square law of the Coulomb force. Both total energy and vector angular momentum are constant for each planet as it revolves about the sun. The possible energy states of each planet is quantized analogous to 1) above, but \( n \) is so large that the separation of energy levels is far too small to be observable. Similarly, the angular momentum of an electron whose orbital quantum number is 2 is on the order of \( 10^{-34} \) joule-sec, and the orbital angular momentum of the earth in the sun's gravitational field is on the order of \( 10^{40} \) joule-sec.

Just as Relativistic mechanics simplifies to Newtonian mechanics for \( \frac{v^2}{c^2} \ll 1 \), quantum mechanics gives the same results as classical physics in the domain where experiment has indicated the later to be valid -- Kepler's laws for the motion of the planets, for example. This so-called "Correspondence Principle" is a feature and requirement of modern physics.

**Emitted Radiations**

The time-independent solution of Schrödinger's equation is a function of position only, so the average position of the electron does not oscillate with time; according to Maxwell's equations, it cannot radiate energy. However, there are solutions to the time-dependant equation where the electron oscillates when it changes from one quantum state to another. Such a solution can emit electromagnetic waves (i.e., light waves) whose frequency is the same as that of the oscillation.

A wave function \( \Psi \) of an electron capable of existing in states \( n \) or \( m \) is a linear combination of the individual wave functions for \( n \) and \( m \):

\[
\Psi = a\psi_n + b\psi_m
\]
Let $aa^*$ be the probability that the electron is in state $n$ and $bb^*$ be the probability that it is in state $m$, then $aa^* + bb^* = 1$ (* denotes complex conjugate). At time $t=0$ the electron is in its normal state $n$, $a=1$ and $b=0$, when it is in its excited state $m$, $a=0$ and $b=1$, and when it is back in its normal state $a=1$ and $b=0$ again. Let $x$ be the position of the electron, and $x_{ave}$ = average value of $x$.

The average position of the electron is given by the equation (the integrals go from $-\infty$ to $+\infty$)

$$x_{ave} = a^2 \int x\psi_n\psi_n^*dx + b^2 \int x\psi_m\psi_m^*dx + 2ab^*\cos\left(\frac{2\pi t(E_m - E_n)}{h}\right)\int x\psi_n\psi_m^*dx$$

From the last term, the frequency of the emitted radiation is $\nu = \frac{E_m - E_n}{h}$. This is the beat frequency from frequencies $E_n/h$ and $E_m/h$, as if the electron had simultaneously existed in both states. If the integral in the last term in the above equation is zero, the intensity of the radiation is zero. In this case the transition is said to be "forbidden;"

$$\int x\psi_n\psi_m^*dx = 0 \quad \text{Forbidden Transition}$$

**Allowed Transitions**

If we evaluate the transition integral for all possible quantum state combinations and for $u = x, u=y$ and $u=z$:

$$\int u\psi_{n,i,m,s}\psi_{n',l',m',s'}^*du$$

We find that the only transitions that allow radiation are those where:

1) Orbital quantum number $l$ changes by +1 or -1 ($l - l' = \pm 1$),
2) Magnetic quantum number $m_l$ does not change or changes by +1 or -1 ($m_l - m_l' = 0, \pm 1$),
3) Electron spin quantum number does not change or reverses its sign ($m_s' = m_s$ or $m_s' = -m_s$)
4) Change in total quantum number from $n$ to $n'$ (allowed for all $n$ and $n'$).

In the presence of a magnetic field $B$, the quantum energy of a state may change. For the hydrogen atom, the quantum number $m_l$ has magnetic energy $E_{m_l} = m_l\frac{eB}{4\pi m}$ where $e$ is the charge of the electron, $h$ is Planck's constant and $m$ is the mass of the electron. The three allowed $m_l$ transitions imply that the single spectral line of an $l$ transition with frequency $\nu_0$ can split into three spectral lines with the frequencies: $\nu_0 - \frac{eB}{4\pi m}, \nu_0$, and $\nu_0 + \frac{eB}{4\pi m}$. This splitting of a spectral line in a magnetic field is called the normal Zeeman Effect, Fig. 2. The splitting has been confirmed with experiments.
The quantum number \( m_s \) describes the so-called spin angular momentum of the electron. In the reference frame of the electron, the proton is revolving about it and thereby creates a magnetic field according to Maxwell's equations. The magnetic field acts on the electron's spin angular momentum to produce an "Internal Zeeman Effect." The result is a splitting of every spectral line into two lines. The effect is very small; the wave length shift is about \( 2 \times 10^{-10} \) meters for an unperturbed wave length of \( 5628 \times 10^{-10} \) meters.

**The Electronic Structures of Atoms**

For atoms having more than one electron, Schrödinger's equation can be solved in an analogous way to the one electron case, i.e., by separation of variables. The complete wave function for \( n \) electrons is approximately (i.e., neglecting electron-electron interactions) the product of the individual wave functions:

\[
\psi(1,2,3,\ldots,n) = \psi(1) \psi(2) \psi(3) \ldots \psi(n)
\]

Since all electrons are identical, the probability density of the system should not change if any two electrons are exchanged. Let \( \psi(2,1,\ldots) \) be the wave function representing the exchanged particles, then:

\[
\psi^*(2,1,\ldots) = \psi^*(1,2,\ldots)
\]

which implies \( \psi(2,1,\ldots) = \psi(1,2,\ldots) \) (Symmetric Case) or \( \psi(2,1,\ldots) = - \psi(1,2,\ldots) \) (Anti-symmetric Case). By considering linear combinations of two identical particles in two quantum states we find that in the symmetric case the particles can exist in the same state, and in the anti-symmetric case the two particles cannot exist in the same quantum state.

In 1925 Wolfgang Pauli analysed the spectral lines of all of the elements. He discovered the Exclusion Principle; particles such as electrons, protons and neutrons moving in a common
force field, such as the Coulomb force from an atomic nucleus, never have exactly the same set of quantum numbers. No two electrons had the same values for all four quantum numbers. In particular, electrons do not arrange themselves in the most stable, minimum energy, configuration.

All electrons with the same value of total quantum number \( n \) are, on the average, about the same distance from the nucleus of the atom, interact with nearly the same electrostatic field strength of the nucleus and have nearly the same quantum energies. They occupy the same "atomic shell." The inner most shell, \( n=1 \), electrons have the largest binding energy (i.e., the external energy required to remove an electron to infinity).

The energy of a particular electron also depends on its value for quantum number \( l \), but the energy dependence on \( l \) is not as great as on \( n \). The electrons in each shell decrease in binding energy with increasing values of \( l \). The Electrons in a particular shell that have the same value of \( l \) occupy the same "subshell." All electrons in the same subshell have about the same energies because the energy differences due to different values of \( m_l \) and \( m_s \) are relatively small.

The exclusion principle limits the number of electrons in any particular subshell: there are \( 2l+1 \) different \( m_l \) values and two \( m_s \) values for each subshell \( l \); which implies that there are at most \( 2(2l+1) \) electrons in subshell \( l \). Since there are a maximum of \( n \) subshells (0 to \( n-1 \)) for any particular shell value \( n \), each shell \( n \) has a maximum of \( 2n^2 \) electrons. For example: For shell \( n = 1 \) there is only 1 subshell, \( l = 0 \) with 2 electrons. For shell \( n= 2 \) there are 2 subshells, \( l=0 \) with 2 electrons and \( l=1 \) with 6 electrons for a total of 8 electrons. For shell \( n=3 \) there are 3 subshells, \( l=0 \) with 2 electrons, \( l=1 \) with 6 electrons and \( l=2 \) with 10 electrons for a total of 18 electrons.

A shell or subshell containing all its possible electrons is "closed." The electrons in an atom with all its shells closed are tightly bound to the nucleus because the positive charge of the nucleus is larger than the sum of the negative charges of the electrons in the inner, shielding shells. And an atom with all closed shells has no net magnetic moment because the orbital and spin angular momentums of the electrons in a closed subshell sum to zero and their charge distributions are symmetric. It does not attract electrons and its own electrons are not easily detached. For example, all the subshells of the inert gases are closed subshells.

The first row of Table 1 (row \( n-l \)) shows the order (left to right) in which the subshells of an atom are filled with electrons; the first number is the total quantum number \( n \) and the second number is the orbital angular momentum quantum number \( l \). The second row gives the maximum number (\( \# \)) of electrons that can be contained in subshell \( l \) for a particular value of shell \( n \). The third row (\( \Sigma \)), gives the cumulative electron total. The symbols for the various filled elements are given in the bottom two rows. Row VIII shows the inert gases,
"elements so inactive that they almost never form compounds with other elements, and their atoms do not join together into molecules like the atoms of other gases." Shown in bold, all have an outermost subshell of 1 closed with 6 electrons, except Helium (He) which has an outermost subshell of 0 closed with 2 electrons.

Table 1  Atoms With All Subshells Closed. Rows: n-l (Shell-Subshell), # (Number of Electrons in Subshell), ∑ (Cumulative electron total), VIII (Inert Gas Symbol), X (Element Symbol).

<table>
<thead>
<tr>
<th>n-l</th>
<th>1-0</th>
<th>2-0</th>
<th>2-1</th>
<th>3-0</th>
<th>3-1</th>
<th>4-0</th>
<th>4-1</th>
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<th>6-1</th>
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<td>6</td>
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<td>6</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>∑</td>
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<td>4</td>
<td>6</td>
<td>2</td>
<td>6</td>
<td>2</td>
<td>10</td>
<td>6</td>
<td>2</td>
<td>14</td>
<td>10</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>VIII</td>
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<td>Kr</td>
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<td>Hg</td>
<td>Ra</td>
<td>Cf</td>
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</table>

Table 2 shows that the very active alkali metals Lithium Li, Sodium Na, Potassium K, Rubidium Rb, and Cesium Cs all have a single electron in subshell 0 of their outermost shell. The outer electron is shielded from the nucleus by the electrons closer to the nucleus so the effective binding force is about the same as 1 proton. An alkali metal tends to lose its outermost electron. It is very chemically active with +1 chemical valence. The ionization energy of an element (i.e., the energy needed to remove an electron from one of its atoms) in electron volts (eV) is shown below each of the alkali metals.

The Halogen non-metals Fluorine F, Chlorine Cl, Bromine Br, Iodine I, and Astatine At are also chemically very active; each has -1 chemical valence. The Halogens all have five electrons in the subshell l=1 of their outermost shell. They tend to acquire an electron to close the subshell. The electron affinity of an element (i.e., the energy released when an electron is added to one of its atoms) in electron volts (eV) is shown below each of the halogen non-metals.

Table 2  Alkali Metals, Halogen Non-Metals, and Inert Gases. Rows: n-l Shell-Subshell, # Electrons in subshell, ∑ Cumulative Electron Total, X Element with its Ionization/Affinity Energy in eV. Inert Gases are shown in bold.

<table>
<thead>
<tr>
<th>n-l</th>
<th>2-0</th>
<th>2-1</th>
<th>3-0</th>
<th>3-1</th>
<th>4-0</th>
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<td>6</td>
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<td>1</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>5</td>
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<tr>
<td>∑</td>
<td>3</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>35</td>
<td>36</td>
<td>37</td>
<td>53</td>
</tr>
<tr>
<td>X eV</td>
<td>Li</td>
<td>5.4</td>
<td>F</td>
<td>3.6</td>
<td>Na</td>
<td>5.1</td>
<td>Cl</td>
<td>3.8</td>
<td>Ar</td>
<td>K</td>
<td>4.3</td>
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</tbody>
</table>

16
For example, it is not surprising that NaCl (i.e., table salt) is so common. The amount of energy needed to remove the outer electron from a sodium atom is 5.1 eV (i.e., Na's ionization energy); \( \text{Na} + 5.1 \text{ eV} \rightarrow \text{Na}^+ + e^- \). When the electron is near a chlorine atom, it is added into the Cl atom's outer shell and 3.8 eV (i.e., Cl's affinity energy) is released; \( \text{Cl} + e^- \rightarrow \text{Cl}^- + 3.8 \text{ eV} \). Bringing the two equations together yields \( \text{Na} + \text{Cl} + 1.3 \text{ eV} \rightarrow \text{Na}^+ + \text{Cl}^- \). The net energy required to form Na and Cl ions is only 1.3 eV.

The electrostatic force brings the two ions together until their electron structures begin to interact, creating a repulsive force. “When the electron structures of the two ions overlap, they constitute a single atomic system rather than separate independent systems.” To obey Pauli’s exclusion principal some electrons must go into higher quantum states thereby increasing the total energy of the molecule. Their quantum states became entangled.

\textit{Entanglement (Verschränkung) and Non-Locality}

In 1801, Thomas Young conducted an experiment wherein light was focused on a plate with two, closely placed, slits. Some distance behind the plate was a detection screen where the light after passing through the slits could be observed. He saw an interference pattern; a central bright line with dark lines on either side, each of the dark lines was followed by an adjacent dimmer lighted line, which was in turn followed by another dark line and so on. This was analogous to the pattern of crests and troughs observed for water waves after passing through two openings in a barrier. He concluded that light was a wave (50 years before Maxwell's wave equation and 150 years after Newton said that light was particles).

Since 1925 similar double-slit experiments have been conducted often, with ever increasing sophistication. Photons, electrons, atoms and even fairly large molecules like a buckyball have been sent through double slits. A buckyball (60 carbon atoms) is large enough to be seen under an electron microscope. Recently, the experiment was preformed with molecules that each comprised 810 atoms (about \( 10^8 \) times the mass of an electron). In all cases the same basic interference pattern was observed on the detector screen.

\textbf{Figure 3} Double Split Experiment
For example, consider the double slit experiment wherein an electron gun fires electrons one at a time Fig. 3. After a time we begin to see a pattern developing on the screen in no predictable order. After many more electrons have been sent, we see that the pattern on the screen is the classical wave interference pattern; the more electrons fired the better the pattern becomes. The higher the concentration of electron hits the lighter the line, and the lower the concentration of electron hits the darker the line. If a detector is placed in front of each slit that measures which particle goes through which slit the diffraction pattern will not form. In fact, the more that is known about the electrons paths, the more the resulting interference pattern is degraded.

A physical electron does not split in two; each passes through one slit or the other. The question is how do all these electrons "know" the trajectories to take in order to create the observed interference pattern? The correct answer is no one knows. "The double-slit experiment...has in it the heart of quantum mechanics. In reality, it contains the only mystery ... nobody can give you a deeper explanation of this phenomenon than I have given; that is, a description of it" [1965 Richard Feynman].

The light lines and dark lines of the interference pattern closely match the peaks and valleys of the electron probability density function $\Psi\Psi^*$ where $\Psi$ is a solution of the Schrödinger equation applied to the double-slit experiment, Fig. 3. The wave function $\Psi$ is the starting point for most interpretations, "the trajectories of a many-body quantum system are correlated not because the particles exert a direct force on one another (e.g., a Coulomb force) but because all are acted upon by an entity – mathematically described by the wave function." Particles without mass, like photons, have no solutions of the Schrödinger equation so have another wave to guide them.

In the de Broglie-Bohm interpretation particles have well-defined, but unknown, initial positions that "evolve according to the 'guiding equation', which expresses the velocities of the particles in terms of the wave function. ... The slit through which each particle passes and its location upon arrival on the photographic plate are completely determined by its initial position and wave function. While each trajectory passes through only one slit, the wave passes through both; the interference profile that therefore develops in the wave generates a similar pattern in the trajectories guided by the wave" (2017 Sheldon Goldstein). In the Copenhagen interpretation particles do not have exactly determined physical properties prior to being measured. In some manner the sensing apparatus selects particular locations to display from the possible locations provided by the wave function. In both interpretations the trajectories of multiple interacting particles become correlated by a single, joint, wave function and both make exactly the same predictions.

In classical physics and special relativity (and general relativity) the universe is "local." For an event to have an effect on another, distant, event it must somehow traverse the space in
between them (e.g., via sound waves, photon beams, a telephone call...). Distant objects are unable to have direct influence on one another; an object is directly influenced only by its immediate surroundings.

In quantum mechanics the universe is non-local; events can affect each other instantaneously, even if they are light years apart. The velocity and acceleration of any one particle depends on the instantaneous positions of all other particles. Mathematical criteria for non-locality derived by John Bell in 1964 and subsequent experiments have proven, many times over, that the results predicted by quantum mechanics cannot be explained by any theory which preserves locality; “spooky actions at a distance” are a feature of quantum mechanics.

Non-locality occurs due to the phenomenon of entanglement, wherein particles that interact with each other become permanently correlated - dependent on each other’s states and properties, to the extent that they effectively lose their individuality and in many ways behave as a single entity. According to Schrödinger, Entanglement (Verschränkung) occurs when two or more particles (e.g., photons, electrons, atoms, molecules...) "enter into a temporary physical interaction due to known forces between them, and when after a time of mutual influence the system separates again. They can no longer be described in the same way as before - by the interaction, their quantum states became entangled."

Consider two electrons, e1 and e2, that have been entangled such that if one has a spin of +1/2 then the other must have a spin of -1/2 (conservation of angular momentum). We give one electron to Rika and the other to DeA. DeA takes her electron and travels to a distant town or even to another planet. Quantum mechanics tells us that the two quantum states have equal probability: [e1-1/2, e2+1/2], [e2-1/2, e1+1/2], and Mother Nature will not decide the actual spin of either electron until one of the girls actually makes a measurement. Let us say that Rika measures the spin of her electron, only then does Mother Nature roll the dice to determine the spin of Rika's electron. If Rika finds that her electron has a spin of 1/2, then DeA will find, instantaneously, that her electron spin is -1/2.

The principle of superposition means that an entangled quantum system exists in many states at the same time. It changes from having many possible quantum states to being randomly found in only one of those possible states when a measurement is made; the wave function "collapses." By analogy, we can think of the two electrons as two coins spinning on a table top; each coin has a head and a tail at the same time. If Rika slaps her coin down (makes a measurement), it will stop spinning and show either a head or a tail. Let's say it shows a head; then immediately DeA's coin will stop spinning and show a tail. A 2015 free-space entanglement-swapping experiment between the Canary Islands of La Palma and Tenerife verified the presence of quantum entanglement between two previously independent photons separated by 143 km.
In 1825 Laplace showed that the solar system would have flown apart long ago unless the speed of gravity was at least 108 times faster than the speed of light. Modern computations show that the speed of gravity is at least $2 \times 10^{10}$ times the speed of light: it takes 8.3 minutes for light to travel from the sun to the earth. If the speed of gravity were equal to the speed of light then the gravitational force between the sun and earth would not be along the line connecting them but rather along a line connecting the earth to where the sun was 8.3 minutes ago. With the resulting torque, the distance between sun and earth would double in about 1200 years. Not surprising, scientists studying celestial mechanics have traditionally needed to assume that all gravitational interactions between moving stars are instantaneous. (1998 Tom Van Flandern)

In the 1600s Sir Isaac Newton said, “That one body may act upon another at a distance through a vacuum, without the mediation of anything else, by and through which their action and force may be conveyed from one to the other, is to me so great an absurdity, that I believe no man who has in philosophical matters a competent faculty of thinking, can ever fall into it.” Newton has received two answers; the first from Albert Einstein in 1915 under the title The General Theory of Relativity (GTR), and the second from Erwin Schrödinger in 1926 under the title Quantum Entanglement.

GTR formally implements the equivalence principal wherein the gravitational mass $m$ of a body that appears in Newton’s law of gravitation ($F = mM_eG/r^2$) is equivalent to the body’s inertial mass $m$ that appears in Newton’s second law ($F=ma$). This means that any accelerating reference frame can be viewed as a constant velocity (inertial) frame by assuming the presence of a suitable gravitational field; Special Relativity (SR) is thereby extended to reference frames that accelerate. And so its postulate, “information cannot travel faster than the speed of light” is extended to all reference frames.

So that the solar system does not then fly apart, Einstein introduces the mathematical concept called “curved space-time.” The curved space-time effect mathematically replaces gravity and cancels out the "information cannot travel faster than the speed of light" effect so that GTR gives essentially the same result as the "infinite speed of gravity” observed in the normal Euclidean space of Newtonian mechanics. For weak gravitational fields (e.g. our sun) and mass speeds much less than the speed of light, GTR converges to Newton’s laws.

Even with its four dimensional matrix equation that is difficult to interpret and nearly impossible to solve, GTR still retains the same fundamental restriction as classical physics; its universe is "local." For an event to have an effect on another, distant, event it must somehow traverse the space in between them.
Unlike GTR, the quantum mechanics universe is non-local; events can affect each other instantaneously, even if they are light years apart. If masses are previously entangled through the action of gravity (e.g., during the big bang or before) they could form a system wherein all mass locations are "known" by all masses all of the time; entangled masses do not need to send out information about their movements.

Today, quantum mechanics is the language of science at the atomic and sub-atomic scales. It has already given us many commercial products, for example: semi-conductors and transistors that drive digital computers, electron microscopes that exploit the wave properties of electrons to show objects many thousands of times smaller than those seen by traditional microscopes, magnetic resonance imaging (MRI) wherein the magnetic resonances are based on the atomic spin of the body's atoms, lasers that rely on photons radiated during electron transitions. Quantum entanglement is the basis for exciting research in quantum teleportation, quantum cryptography, super dense computing, and quantum communication channels between distant users.

Quantum non-locality and entanglement raise old and new philosophical questions. Are all individuals somehow quantum entangled in a non-local universe, marching through life's "double slits" with no apparent rhyme or reason until all our yesterdays unveil God's celestial plan/wave function? How would Hegel respond to the new triad of non-locality, entanglement and chance? Is "The Power of Now" a macroscopic realization of quantum entanglement?

**Nuclear Forces**

The fourth veil was lifted, but not removed, during the second quarter of the 1900s. What is the structure of an atom's nucleus? What keeps the densely packed, positively charged, protons in an atom's nucleus from flying apart due to the repulsive coulomb forces between them? What is the size and weight of an atom's nucleus? Why do uranium salts leave well defined streaks and lines when placed on a photographic plate? Why do some atoms naturally decay into other atoms? Why does splitting an atom release energy? How do nuclear power plants work? How does the sun generate the electromagnetic radiation that heats the planets?

In 1932 James Chadwick identified the mysterious radiation found in the experiments of Bothe and Becker. He determined that the nucleus of an atom consisted not only of protons, but also particles which he called "neutrons." A neutron has a mass of $1.675\times10^{-27}$ kg, slightly greater than the mass of a proton and about 2000 times greater than the mass of an electron. It has no electrical charge, and can only exist outside a nucleus as a free particle for about 1000 seconds before decaying into a proton and an electron.
The common symbology for describing an atom's nucleus is \( _Z^AX \) where \( X \) equals the chemical symbol of the element, \( Z \) equals the number of protons, and \( A \) equals the number of neutrons plus the number of protons. Examples are: \(_{92}^{238}\text{U} \), \(_{8}^{16}\text{O} \), \(_{2}^{4}\text{He} \), and \(_{1}^{1}\text{H} \).

The volume of a nucleus is proportional to \( A \) which implies that the radius \( R \) of a nucleus is proportional to the cube root of \( A \); \( R = R_0A^{1/3} \) where \( R_0 \) is determined experimentally to be \( R_0 \approx 1.2 \times 10^{-15} \) meters. The density of nuclear matter is \( 2 \times 10^{17} \) kg/m\(^3\) (a billion tons per cubic inch); about the "density of collapsed, white dwarf, stars."

A stable nucleus has a smaller mass than the sum of the separate free-state masses of the neutrons and protons that make up the nucleus. For example, the mass of a lithium nucleus \(_{3}^{6}\text{Li} \) is less than the total mass of the three free neutrons and three free protons of which it is composed. The missing mass, \( \Delta m = 0.05715 \times 10^{-27} \) kg, is equivalent to an energy of 32.1 MeV according to Special Relativity. To break up a lithium nucleus into free neutrons and protons requires 32.1 MeV of energy from an outside source. This is called the binding energy of the nucleus.

The binding energy per nucleon versus mass number \( A \) increases steeply from \( A = 1 \) (\(_{1}^{1}\text{H} \), Hydrogen) until \( A = 56 \) (\(_{26}^{56}\text{Fe} \), Iron) and then drops slowly Fig. 4. The figure suggests that energy will be set free if a heavy nuclei splits into two lighter, more stable, nuclei (i.e., nuclear fission), or if lighter nuclei combine into a more stable, heaver, nucleus (i.e., nuclear fusion). In either case the difference in mass will be released as energy.

The so-called Nuclear Strong Force holds protons and neutrons together into a nucleus. The force acts between each of the three possible combinations of nucleons: proton-neutron, neutron-neutron and proton-proton. It is an attractive force for separation distances from about \( 0.7 \times 10^{-15} \) meters to about \( 4 \times 10^{-15} \) meters (about the size of a nucleus) and then falls...
off rapidly to zero. At separations smaller than about $0.7 \times 10^{-15}$ meters, the strong force becomes repulsive and stops the nucleus from collapsing on itself. This repulsive component is responsible for the physical size of the nucleus since the nucleons can come no closer than the strong force allows.

Over its very limited range, the Nuclear Strong Force is more than 100 times stronger than the electromagnetic Coulomb force, $10^{38}$ times stronger than the gravitational force, and about a million times stronger than the so-called "Weak Force." The weak force is responsible for radioactive beta decay by converting a neutron into a proton and an electron. The range of the weak force is less than the diameter of a proton.

**Nuclear Decay**

The strong force overcomes the electromagnetic proton-proton repulsive forces until the nucleus contains more than 83 protons, $^{83}\text{Bi}^{209}$. Elements with more than 83 protons are unstable; the strong force is barely able to counterbalance the repulsive forces of its protons. They can decay spontaneously. They are said to be radioactive, uranium $^{92}\text{U}^{238}$ for example. The nucleus of a radioactive atom decays by radiating an alpha, beta and/or gamma particle.

An alpha particle consists of one helium nuclei, $^2\text{He}^4$. A beta particle is an electron emitted by a neutron. A gamma particle is a photon emitted by an excited nucleus. After a radioactive nucleus decays by the emission of an alpha or beta particle, the resulting nucleus is normally left in a higher energy "excited" state. It decays back to a lower energy state by emitting a gamma ray photon. Radioactive decay is normally accompanied by a range of X-rays; an emitted alpha particle can displace orbital electrons allowing electrons in higher orbits to emit x-rays as they transition down to the vacated orbits (Table 3).

<table>
<thead>
<tr>
<th>Radiation</th>
<th>Description</th>
<th>Source</th>
<th>Speed $c$</th>
<th>Charge</th>
<th>MeV</th>
<th>Grams</th>
</tr>
</thead>
<tbody>
<tr>
<td>alpha α</td>
<td>$^2\text{He}^4$</td>
<td>nucleus</td>
<td>0.03-0.07</td>
<td>2</td>
<td>4.9</td>
<td>$6.7 \times 10^{-24}$</td>
</tr>
<tr>
<td>beta β</td>
<td>electron e⁻</td>
<td>nucleus/neutron</td>
<td>0.08-0.99</td>
<td>-1</td>
<td>0.003-0.13</td>
<td>$9.1 \times 10^{-28}$</td>
</tr>
<tr>
<td>gamma γ</td>
<td>photon</td>
<td>excited nucleus</td>
<td>1</td>
<td>0</td>
<td>0.03-0</td>
<td>0</td>
</tr>
<tr>
<td>x-ray</td>
<td>photon</td>
<td>electron transition</td>
<td>1</td>
<td>0</td>
<td>$10^{-4}$-.02</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $N$ equal the number of nuclei of a radioactive material present in a sample at a certain time $t$. Let $R$ equal the rate at which $N$ decreases at this particular time (i.e., disintegrations
per second at time $t$). Experiments indicate that $R$ always decreases exponentially with time. By "half life" we mean the time $t_{\text{hl}}$ it takes for $R$ to go to $R/2$. Experiments also show that half life is constant for each particular radioactive element (i.e., radioisotope); at any time $t$, half of the remaining nuclei will live longer than $t_{\text{hl}}$ and half will live shorter than $t_{\text{hl}}$.

Depending on the radioisotope, half life can be a millionth of a second or billions of years. For example, the half life of Uranium 238 decaying to Thorium 234 is $4.51 \times 10^9$ years. This means that each nucleus in a sample of Uranium 238 has a 50% chance of decaying to Thorium 234 (by emitting an alpha particle) in a period of 4.51 billion years, a 75% probability of decaying in 9.2 billion years, and a 87.5% probability of decaying in 13.53 billion years. The mean lifetime of an isotope whose half life is $T_{1/2}$ is $1.44T_{1/2}$; the mean lifetime of Uranium 238 is 6.4944 billion years --if you add up all the individual lifetimes of the nuclei and divide by the total number of nuclei you get their mean or average lifetime.

Radioisotopes continue decaying until they reach a stable end product. The products of the Uranium series as it decays from $^{238}_{92}\text{U}$ to its stable end product lead $^{206}_{82}\text{Pb}$ are shown in table 4. The first row shows, in the order of their creation (from left to right), each isotope's element symbol, the number of its nucleons and the number its protons. The second row shows the type of radiation released by the radioisotope; $\alpha$, $\beta$, $\gamma$. The third row gives the characteristic half life of each radioisotope.

<table>
<thead>
<tr>
<th>$^A_X\text{Z}$</th>
<th>Radiation Type</th>
<th>Half Life</th>
</tr>
</thead>
<tbody>
<tr>
<td>U 238</td>
<td>Th 234</td>
<td>Pa 234</td>
</tr>
<tr>
<td>92</td>
<td>90</td>
<td>91</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\beta$, $\gamma$</td>
<td>$\beta$, $\gamma$</td>
</tr>
<tr>
<td>$4.51 \times 10^9$ yrs.</td>
<td>24.5 days</td>
<td>1.14 min.</td>
</tr>
</tbody>
</table>

Radioactive decay is the principle behind the dating of materials such as a rock or bone; the amount of a radioactive impurity left over in the rock or bone from when it was formed is compared to the amount of the impurity's decay products found in the rock or bone. Age is estimated based on the known half-lives.

There is no classical explanation of how an alpha particle can actually escape from a nucleus. To escape requires energy of about 25 Mev, but alpha particles have energies between only 4 and 9 Mev. However, solutions of the Schrödinger equation, $\psi\psi^*$, show a finite probability that a particle will penetrate the barrier after many tries, and they correctly predict the wide variations in half life shown in table 4. In classical physics there is
no chance of an alpha particle escaping from its atom but in quantum mechanics there is a small but finite chance.

Other than spontaneous decay, there are two basic kinds of nuclear reactions; fusion where two small nuclei combine, and fission where a heavy nucleus (A>230) is split into fragments.

**Nuclear Fission**

Nuclear fission was discovered in 1938 by the German nuclear chemist Otto Hahn and his assistant Fritz Strassmann. When they fired neutrons at Uranium atoms they found the atoms split into two pieces. The sum of the weights of the pieces weighed less than the original atoms. The difference in weight had apparently been converted into kinetic energy of the pieces and subsequently heat in accordance with Einstein’s $E = mc^2$.

In the Fission of $^{92}_{235}U$ a neutron strikes, and is absorbed by, a $^{92}_{235}U$ nucleus forming $^{92}_{236}U$ which in turn splits into the elements $^{36}_{92}Kr$ and $^{141}_{56}Ba$. Because heavier nuclei have a higher neutron/proton ratio than and do lighter nuclei, the nuclei of the two fission fragments contain an excess of neutrons. To reduce the excess, neutrons are emitted by the fragments as soon as they are formed. Subsequent beta decays wherein a neutron in the nucleus becomes a proton by emitting an electron also helps to bring the neutron/proton ratios of each fragment nucleus to a stable value.

![U-235 Fission Chain Reaction](image)

**Figure 5  U-235 Fission Chain Reaction**

Nuclear Fission is the basis for nuclear power plants and nuclear weapons. Splitting one $^{235}U$ nucleus releases several million times more energy than exploding one molecule of dynamite. The basic idea is to get a chain reaction going; a split $^{235}U$ nucleus causes the release of three neutrons that in turn each split other $^{235}U$ nuclei. The number of split $^{235}U$ nuclei will increase geometrically: 1-3-9-27-... $3^n$, Fig 5.
A problem is that natural uranium is 99.3% \textsuperscript{238}U and only 0.7% \textsuperscript{235}U. Neutrons released by splitting \textsuperscript{235}U have a high enough speed (1MeV) to split the more plentiful \textsuperscript{238}U atoms; thereby removing most of the neutrons needed by \textsuperscript{235}U to sustain a chain reaction (splitting \textsuperscript{238}U will not result in a chain reaction). On the other hand, even a slow speed, so-called thermal neutron (1eV) will be absorbed by \textsuperscript{235}U and split it.

The basic idea of a fission reactor is shown in Fig. 6. A \textsuperscript{235}U nucleus is bombarded with a neutron. The resulting \textsuperscript{236}U produced is unstable and splits into two fragments releasing energy, mostly kinetic energy of the fragments that ultimately heat the water in the reactor. The high speed neutrons released by the fragments are modulated, i.e., slowed down to thermal speed by the water in the reactor, so that they can only be absorbed by other \textsuperscript{235}U atoms, thereby maintaining the chain reaction. Control of the speed of the chain reaction is achieved by inserting/removing rods made of cadmium or boron in the water that absorb slow neutrons and thereby slow down/speed-up the reaction to the desired rate.

![Figure 6 Nuclear Fission Reactors](image)

**Table 5 Characteristics of Reactor Products**

<table>
<thead>
<tr>
<th>Products</th>
<th>Energy %</th>
<th>Range cm</th>
<th>Delay</th>
</tr>
</thead>
<tbody>
<tr>
<td>\textsuperscript{36}Kr\textsuperscript{22} , \textsuperscript{141}Ba\textsuperscript{41}</td>
<td>80</td>
<td>&lt; .01</td>
<td>prompt</td>
</tr>
<tr>
<td>Fast Neutrons</td>
<td>3</td>
<td>10-100</td>
<td>prompt</td>
</tr>
<tr>
<td>Gamma γ</td>
<td>4</td>
<td>100</td>
<td>prompt</td>
</tr>
<tr>
<td>Beta Decay β</td>
<td>4</td>
<td>&lt; 10</td>
<td>delayed</td>
</tr>
<tr>
<td>Neutrinos\textsuperscript{ν}</td>
<td>5</td>
<td>∞</td>
<td>delayed</td>
</tr>
<tr>
<td>Neutron Capture</td>
<td>4</td>
<td>100</td>
<td>delayed</td>
</tr>
</tbody>
</table>

\textsuperscript{ν} is the symbol for neutrino; a chargeless, massless particle with spin of 1/2. (Table Source: 2017 Energy from Nuclear Fission, INFN Genoa)

"A 1 GW thermal power reactor provides 10^9 Joule/sec. Each fission produces 200 MeV = 3.2x10^11 joule, and there are 3x10^19 fissions per second (i.e., 3x10^19 \textsubscript{92}U\textsuperscript{235} nuclei disappear per second) which is about 12mg/sec or 300 kg per year. The volume of 300 kilograms of
\( \text{U}^{235} \) is a cube with 25cm per side. To generate the same amount of thermal power would require about 700 million cubic meters of methane gas per year or 700 million liters of oil per year or 1 million metric tons of coal per year." (2017 M. Ripani)

**Fusion**

The fusion of hydrogen into helium is the main source of energy in the universe. The fusion problem is to overcome the coulomb repulsive forces while bringing two positively charged nuclei close enough together to where the nuclear strong force dominates and holds the protons together. So far only the sun's core offers the required environment needed to produce a practical fusion cycle.

The temperature in the sun's core is millions of degrees Celsius; protons have enough kinetic energy to overcome the coulomb repulsive forces. The proton-proton cycle is described below: \( \nu \) is the symbol for neutrino, \( e^+ \) is the symbol for a positron, a particle the mass of an electron but with a positive charge, and \( \gamma \) is the symbol for gamma radiation. Each cycle releases \( \Delta mc^2 = 24.7 \) MeV, where \( \Delta m \) is the difference between the mass of four protons and the mass of an alpha particle plus two positrons.

1-1) \( _1^1 \text{H} + _1^1 \text{H} \rightarrow _1^2 \text{H} + e^+ + \nu 
1-2) \ _1^1 \text{H} + _1^1 \text{H} \rightarrow _1^2 \text{H} + e^+ + \nu

2-1) \ _1^1 \text{H} + _2^2 \text{He} \rightarrow _2^3 \text{He} + \gamma 
2-2) \ _1^1 \text{H} + _2^2 \text{He} \rightarrow _2^3 \text{He} + \gamma

3) \ _2^3 \text{He} + _2^3 \text{He} \rightarrow _2^4 \text{He} + _1^1 \text{H} + _1^1 \text{H}

Each cycle consumes more Hydrogen than it produces - for the past 4.5 Billion years. Most predictions are that the sun will continue to shine as it has for at least another billion years.

END

Note: Correction on page 26, .07 to .7, uploaded on 12 Oct 2020.